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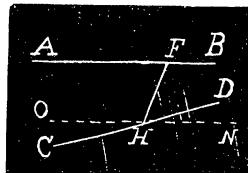
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PROPOSITION XI. *If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.* Euclid, Axiom 12.

PROOF. Let the straight line FH meet the two straight lines AB and CD , making the two angles BFH and FHD together less than two right angles, then AB and CD shall meet, if continually produced, on that side of FH towards B and D . Since the angles BFH and AFH are together equal to two right angles, they must be greater than the sum of the two angles BFH and FHD . Therefore, the angle AFH must be greater than the angle FHD . Hence, draw the line ON through H making the angle FHN equal to the angle AFH (I, 23). Then ON is parallel to AB (I, 27). Therefore CD cannot be parallel to AB (i), and therefore CD and AB must meet if sufficiently produced. Since the sum of the angles AFH and FHO equals two right angles (j), the sum of the angles AFH and FHC must be greater than two right angles. Therefore AB and CD cannot meet on that side of FH toward A and C for then we should have a triangle the sum of whose angles would be greater than two right angles which is impossible by (h). Therefore they must meet on that side of FH toward B and D .
Q. E. D.



ZERO, INFINITESIMALS, INFINITY, AND THE FUNDAMENTAL SYMBOL OF INDETERMINATION.

By GEORGE LILLEY, Ph. D., Professor of Mathematics, State University, Washington.

The following is an outline of the method I use in explaining to the student in algebra how zero is used as a multiplier and a divisor, and how infinitesimals and infinity are used as divisors; also, interpretations of the results obtained by their use.

If we multiply a by a number that decreases by 1 each time beginning with any number, as $+4$, and continue the multiplication until -4 is reached, each product will decrease by a . Thus,

a	a	a	a	a	a	a	a
$+4$	$+3$	$+1$	zero	-1	-2	-3	-4
$+4a$	$+3a$	$+a$	zero,	$-a$	$-2a$	$-3a$	$-4a$,

where zero is a constant number and obtained by subtracting any number from itself, as, $a-a=0$, 0 representing absolute zero.

Evidently a multiplied by zero is one a less than a multiplied by $+1$, or

$a \times \textcircled{1} = \textcircled{1}$; also, a multiplied by -1 is one a less than a multiplied by zero, or $a \times -1 = -a$. Similarly $a \times -2 = -2a$, $a \times -3 = -3a$, etc.

Hence, *If a constant number be multiplied by zero, the product is zero.*

Division may be defined as the process of finding how many times the divisor can be subtracted from the dividend and leave zero.

Dividing 12 by a number that decreases by unity each time, beginning with $+3$, we have

$$\frac{12}{+3} = 4, \quad \frac{12}{+2} = 6, \quad \frac{12}{+1} = 12, \quad \left(\frac{12}{\text{zero}} \right), \quad \frac{12}{-1} = -12, \quad \frac{12}{-2} = -6, \quad \frac{12}{-3} = -4; \text{ etc.}$$

The quotient 4 means that only 3 times $+4$ can be subtracted from 12 and leave zero; and so on for the other quotients.

Since the divisor decreases by unity, the divisor one less than 1 is zero. The divisors less than zero are -1 , -2 , -3 , etc., respectively. Then, the quotient, when zero becomes the divisor, must be between the quotients given by taking $+1$ and -1 as divisors, or between $+12$ and -12 .

Then $\frac{12}{\text{zero}}$ or $\frac{12}{\textcircled{1}} = \ominus$, where $\ominus$ represents no number of times. That is, there is *no number of times zero* that the divisor, zero, can be subtracted from 12 and *leave absolutely nothing*.

Since negative numbers are less than zero, $\textcircled{1}$ is not the least divisor of 12, or of any other number. If $\frac{12}{\textcircled{1}} = \text{infinity}$, or the *largest possible number*, 12 divided by -1 can not give -12 for a quotient. If $\frac{12}{-1} = -12$, $\frac{12}{\text{zero}}$ or $\frac{12}{\textcircled{1}}$ can not give infinity for a quotient, for the divisor, -1 , is one less than the divisor $\textcircled{1}$.

$$\text{Hence, in general, } \frac{a}{\textcircled{1}} = \ominus.$$

For the quotient, $\ominus$, means that there is no number of times zero that the divisor, $\textcircled{1}$, can be subtracted from a and leave zero.

Hence, *If a constant number be divided by zero, the quotient is no number of times.*

It is a consequence of confounding the 0, arising from dividing a by *infinity*, with the absolute zero, that so much confusion has been created in the discussions on this subject. All absolute zeros are constants. The other 0's, used in these discussions, are infinitesimals and variables, and may be less than $\textcircled{1}$.

Since an infinitesimal can be subtracted from a an infinite number of times and leave zero; therefore, $\frac{a}{\textcircled{\ominus}} = \infty$, where $\textcircled{\ominus}$ represents an infinitesimal.

That is, *If a constant number be divided by an infinitesimal, the quotient is infinity.*

Suppose, for illustration, we divide a by a number that diminishes one-

tenth each time, beginning with one ; we will have the series

$$\frac{a}{1}=a, \quad \frac{a}{\frac{1}{10}}=10a, \quad \frac{a}{\frac{1}{100}}=100a, \quad \frac{a}{\frac{1}{1000}}=1000a, \quad \frac{a}{\frac{1}{10000}}=10000a, \quad \dots\dots$$

Evidently, by continuing the series indefinitely, the divisor becomes *less* than any assignable number however small, and the value of the quotient increases without limit and becomes *greater* than any assignable number however great.

Hence, *If a constant number be divided by a decreasing variable, as the variable becomes too small to be expressed, the quotient becomes too large to be expressed.*

Since infinity can be subtracted from a the infinitesimal part of once and leave zero ; therefore, $\frac{a}{\infty}=\odot$.

That is, *If a constant number be divided by infinity, the quotient is infinitesimal.*

Suppose the divisor, in the above illustration, increases each time, beginning with 1 ; we will have the series

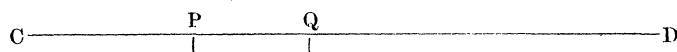
$$\frac{a}{1}=a, \quad \frac{a}{10}=.1a, \quad \frac{a}{100}=.01a, \quad \frac{a}{1000}=.001a, \quad \frac{a}{10000}=.0001a, \quad \dots\dots$$

Evidently, by continuing the series indefinitely, the divisor becomes greater than any assignable number, however great, and the value of the quotient decreases without limit and becomes *less* than any assignable number, however small.

Hence, *If a constant number be divided by an increasing variable, as the variable becomes too great to be expressed, the quotient becomes too small to be expressed.*

This subject is also illustrated in interpreting the results obtained by assigning different values for the rates of travel and the distance apart, in Clairaut's problem, of the Couriers.

"Two Couriers, A and B, travel in the same direction, CD , at the rates m and n miles an hour, respectively. If at any time, say 12 o'clock, A is at P , and B is at Q , a miles from P , at what time and at what place are they together ?"



Let t =the number of hours traveled, after 12 o'clock, to the place where A overtakes B, and d =the number of miles travelled by A in t hours ; or the number of miles from P to the place where A overtakes B.

Since the number of miles travelled by each, after 12 o'clock, equals the rate multiplied by the number of hours, we have

$$d=mt, \text{ and } d-a=nt.$$

Solving these equations, we have

$$t = \frac{a}{m-n}, \quad d = \frac{am}{m-n}.$$

We will now examine these values under different conditions.

1. If $m > n$.

This condition makes the values of t and d *positive*. That is, the Couriers are together *after* 12 o'clock, and at some place to the *right* of P .

If $m < n$.

This condition makes the values of t and d *negative*. That is, the Couriers are together *before* 12 o'clock, and to the *left* of P . This interpretation corresponds with the conditions made. For, if m is less than n , A travels more slowly than B , and it follows that they must have been together before 12 o'clock, and before they could have advanced as far as P .

3. If $m = n$, or $m - n = \text{zero}$.

Then $t = \frac{a}{\textcircled{1}}$, and $d = \frac{am}{\textcircled{1}}$.

As there is no number of times zero that subtracted from a leaves zero, there is no number of hours when they have been or will be together. Furthermore, as every number of times zero subtracted from a leaves a ; that is, $a - v \times \textcircled{1} = a$, where v represents any number whatever, they are always the same distance apart.

Hence, A result in the form $\frac{a}{\textcircled{1}}$ indicates that the problem is impossible.

This interpretation corresponds with the supposition made. For, if m is equal to n , the Couriers travel at the same rate, and since they were a miles apart at 12 o'clock, it is evident they never could have been, and never will be, together.

4. If $a = \text{zero}$, and $m > n$, or $m < n$.

Then $t = \frac{\textcircled{1}}{m-n}$, and $d = \frac{\textcircled{1}}{m-n}$.

They are together at the start, as shown by $a = \text{zero}$; but, as there is no number of times $m - n$ that subtracted from zero, will leave zero, they can never be together again.

Furthermore, the longer the time, the greater or less will $m - n$ be; hence, they will be constantly diverging. For example,

In 1 hour, $\text{zero} - (m - n) = n - m$, distance apart;

In 2 hours, $\text{zero} - 2(m - n) = 2n - 2m$, distance apart;

In 3 hours, $\text{zero} - 3(m - n) = 3n - 3m$, distance apart, etc.

Hence, $t = \frac{\textcircled{1}}{m-n}$ indicates that they will be together in zero hours after 12

o'clock, but never after or before. For $\text{zero} - \textcircled{1} \times (m - n) = \text{zero}$, is the only value for t that will satisfy the conditions.

Similarly, $d = \frac{\textcircled{1}}{m-n}$ means no distance from P , and shows that they were placed together by the conditions of the problem, $a = \text{zero}$, but for *all other time the problem is impossible*.

5. If $a = \text{zero}$, and $m = n$.

Then $t = \frac{\textcircled{1}}{\textcircled{1}}$, and $d = \frac{\textcircled{1}}{\textcircled{1}}$.

As any number of zeros subtracted from zero gives zero ; that is, $\text{zero} - v \times \textcircled{1} = \text{zero}$, where v represents any number whatever, they are together at all times ; for $t = \text{any number}$.

Hence, $t = \frac{\textcircled{1}}{\textcircled{1}}$ means all conceivable times, and $d = \frac{\textcircled{1}}{\textcircled{1}}$ means all conceivable distances, and are indeterminate, not being one, but every value.

Therefore, *A result $\frac{\textcircled{1}}{\textcircled{1}}$ indicates that the problem is indeterminate.*

The form, $\frac{\textcircled{1}}{\textcircled{1}}$, is a *symbol of indetermination*, and does not indicate that no solution can be found, but that too many can be determined. The indetermination consists in the fact that any one of the infinite solutions will answer just as well as any other.

January 11, 1897.

EDITORIALS.

This issue of the MONTHLY was delayed on account of securing sorts for Dr. Lovett's article.

Prof. E. L. Brown, formerly of the Capital University, Columbus, Ohio, is now a member of the Faculty of the Department of Mathematics of the Colorado State University.

The articles on "Euclidean Geometry Without Disputed Axioms," and "Zero, Infinitesimals, Infinity, and the Fundamental Symbol of Indetermination," are published at the request of the authors. They invite criticism on their respective articles, and, if there is any defect in the reasoning by which they arrived at their conclusions, they desire to have the same pointed out.

BOOKS AND PERIODICALS.

A Text-Book of Light. With Numerous Diagrams and Examples. By R. Wallace Stewart, D. Sc., London, Author of "An Elementary Text-Book of Heat and Light," "An Elementary Text-Book of Magnetism and Electricity," etc.